It has been suggested that the properties of "integration" and "differentiation" are necessary for the emergence of consciousness. We present a dynamical system model that is based on these two conditions. The collection of neurons are partitioned into clusters on which we define a map that reflects the communication between clusters. Such a map displays the forward and backward circuitry between clusters in a probabilistic manner. The presence of "re-entry" guarantees that the map is sufficiently complex, that is, nonlinear and chaotic, to possess numerous invariant sets of clusters, which are referred to as agglomerations. We suggest that an agglomeration that is mixing characterizes a conscious state. The model establishes a theoretical framework that may structure and encourage experimental work.

Keywords: Consciousness; clusters of neurons; expanding dynamical system; invariant sets of clusters; invariant measures.

1. Introduction

Neurons form groups which fire together. We refer to such a group as a cluster and consider clusters to be the basic operating units of the brain. Re-entrant connections bind these clusters via parallel reciprocal connections, thereby coordinating information from the different senses to provide a coherent experience of consciousness. In [Tononi & Edelman, 1998] two neural processes are described that might account for conscious experience. First a conscious event is highly integrated; it is a unified experience which cannot be decomposed into smaller meaningful experiences. This unity of consciousness is evinced by the fact that we cannot be conscious of two incongruent scenes simultaneously. It has been suggested that fast integration is achieved by the process of re-entry, “the ongoing recursive, highly parallel signaling within and among brain areas” [Tononi & Edelman, 1998; Edelman & Tononi, 2001]. Experimental evidence (see references in [Tononi & Edelman, 1998]) indicate that integration of distributed neuronal clusters via re-entry is a necessary condition for consciousness. The second property of consciousness is its complexity (differentiation). The number of different conscious states that can be attained in a short period of time is enormous. Stated differently, there is a tremendous amount of information available to the conscious process. Consciousness, then, is a dynamical process on the collection of neurons that possess the properties of local unity together with exceptional differentiation.

It is the objective of this note to present a mathematical model that rigorously captures the foregoing two conditions. The second property follows directly from the richness of the underlying neuronal space. The first property, which is a dynamic condition, is reflected in the nonlinear chaotic map that arises from the communication properties of the clusters. This information is collected in a 0-1
matrix which we call the communication matrix. It is well-known that non-negative matrices can be written in normal block form [Berman & Plemmons, 1979], where each block is either irreducible or a $1 \times 1$ null matrix. In fact, the blocks correspond to maximal sets of communicating clusters. Two clusters are said to communicate if each cluster is accessible from the other. An irreducible submatrix is identified with a communicating set of clusters (agglomerations) and behaves independently of other agglomerations of clusters, yielding locally independent functional behavior. To ensure consciousness we want an agglomeration to achieve a dynamical resonance which is a consequence of the existence of a physical invariant measure on the agglomeration. This measure comes into being (usually very quickly) via the recursive process of re-entry and is described mathematically by the Birkhoff Ergodic Theorem, which provides a weighting for each of the clusters in the agglomeration that achieves local resonance. Dynamically, the irreducibility condition on the submatrix is equivalent to mixing of the clusters in an agglomeration. If a block on the diagonal of the communicating matrix in normal block form consists of $1 \times 1$ nonzero entries on the diagonal only, the collection of these can be considered to form an invariant set, but there is no communication between the clusters. This may correspond to a seizure where there is lots of brain activity but no consciousness. The clusters are not ergodic, like instruments in an orchestra playing independently, displaying much activity and creating sounds but lacking harmony.

The fact that the brain does not behave like a digital computer [Edelman & Tononi, 2001] is inherent in our model since connections are defined only between clusters rather than individual neurons. In fact, in our definition of the communication matrix what is needed is merely the probability of a cluster of neurons turning on other clusters, where turning on may mean in an average sense.

It is of interest to note that in [Tononi & Sporns, 2003; Tononi, 2004] an information theoretic method is developed which describes consciousness and how it may be measured.

2. Model for Consciousness

It is now common knowledge that even simple one-dimensional maps have the ability to describe very complicated dynamical behavior of biological and mechanical systems. Modeling dynamics by a map offers much more variety of behavior than do differential equations whose solutions are greatly restricted by time continuity. Once a map is determined, the long term statistical behavior is described by a probability density function (pdf), which can be obtained by the measurement of the system or by mathematical means using the Frobenius-Perron operator [Boyarsky & Góra, 1997] which transforms a pdf into a pdf under the transformation that takes clusters to clusters. If this map is piecewise expanding [Boyarsky & Góra, 2006], then ergodic dynamical behavior in the form of a pdf (ergodic invariant measure) is assured. Furthermore, we get a partition of the space into a finite collection of agglomerations [Boyarsky & Góra, 1997], each one of which, we claim, corresponds to a conscious experience.

Since any cluster can turn on many other clusters, it is reasonable to postulate that the map $T$ that describes communication between clusters is “piecewise expanding” [Boyarsky & Góra, 1997; 2006; Boyarsky et al., 1997]. The expanding property guarantees complex, ergodic behavior which can be described analytically via the fixed points of the Frobenius–Perron operator. In particular, we get partition of the space into ergodic sets [Boyarsky & Góra, 1997].

We will model the set of neurons with an interval $I = [0, 1]$, a collection of infinitely many points which approximates the enormous number of neurons in the brain. Each $x$ in $I$ corresponds to a neuron. The dynamics of the neurons will be modeled by a transformation $T : I \rightarrow I$.

We will construct a map $T$ in such a way that it has a very large number of ergodic components. A subset $A \subset I$ of $I$ is an ergodic component if and only if for any $x \in A$ all iterates $T^nx$, $n \geq 0$ are also in $A$ and it cannot be divided into two subsets with the same property. Moreover, for any such ergodic component $A$, our $T$ will have an invariant measure $\mu_A$ supported on $A$. The invariant measure $\mu_A$ can be understood as a frequency with which almost every trajectory $\{T^nx\}_{n \geq 0}$, $x \in A$, visits subsets of $A$, i.e.

$$\mu_A(E) = \lim_{n \to \infty} \frac{1}{n} \# \{0 \leq k \leq n - 1 : T^k x \in E\},$$

$E \subset A$.

The well developed theory of such transformations and their invariant measures can be found in [Boyarsky & Góra, 1997].
The construction goes as follows. We divide \( I \) into a large number \( N \) of subintervals \( \{J_k\}_{k=1}^N \) (for convenience, we make them all of equal length — but this is not necessary). Each \( x \in I \) corresponds to a neuron and a subinterval in \( I \) symbolizes a cluster of neurons. The clusters are grouped into agglomerations. To reflect this in the model we divide all subintervals \( \{J_k\}_{k=1}^N \) into \( K < N \) families \( A_k \), \( k = 1, 2, \ldots, K \). The number of subintervals in a family \( A_k \) will be denoted by \( q(k) \) and we have

\[
A_k = \{J_{q(1)}, J_{q(2)}, \ldots, J_{q(k)}\}, \quad k = 1, 2, \ldots, K,
\]

where \( \sum_{k=1}^K q(k) = N \) and each \( J_{q(k)} \) is one of the subintervals \( \{J_k\}_{k=1}^N \).

The map \( T \) is defined as follows: Let us consider interval \( J^k \in A_k \). We divide it into \( q(k) \) equal subintervals and each of them is piecewise linearly mapped onto a different subinterval in the family \( A_k \). This property makes the map \( T \) a semi-Markov map of an interval [Boyarsky & Góra, 1997].

An important property of these maps is that their invariant densities are constant on the intervals \( \{J_k\}_{k=1}^N \). Moreover, map \( T \) has the following important properties:

1. \( T \) has \( K \) ergodic absolutely continuous measures \( \mu_k \), \( k = 1, 2, \ldots, K \). Each \( \mu_k \) is supported on the subintervals of the family \( A_k \).
2. Small changes in the initial point may cause the map to move to different ergodic sets, i.e. different invariant measures will be exposed.
3. \( T \) is piecewise expanding which is a manifestation of the fact that each neuron is connected to many other neurons.

The map \( T \) can be viewed as a map not on points but on intervals. Then, each interval \( J_i \) is transformed onto each of the intervals in the same family \( A_k \). The \( N \times N \) matrix \( C_T \) with entries

\[
C_{ij} = \begin{cases} 
1, & \text{if } J_j \subset T(J_i); \\
0, & \text{otherwise},
\end{cases}
\]

is called the communication matrix of \( T \). We can consider the matrix \( C_T \) instead of the map \( T \). In particular, the communication classes of \( C_T \) correspond to ergodic components of \( T \). If all intervals \( \{J_k\}_{k=1}^N \) are of equal length and we replace 1’s in the communication matrix by the reciprocals of the number of elements in appropriate agglomeration (i.e. replace nonzero \( C_{ij} \) by \( 1/q(k) \) if \( J_i \) and \( J_j \) are in \( A_k \)) the matrix becomes the Frobenius-Perron matrix of \( T \) [Boyarsky & Góra, 1997, Chapter 9]

An Ergodic Theory of Consciousness
May 15, 2009 16:26 02373

A. Boyarsky & P. Góra

Fig. 3. The histograms of trajectories starting from $x_0 = 0.039999$ (dark gray) and $x_0 = 0.040001$ (gray), respectively.

In our example the communication matrix $C_T$ is:

$$
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 
\end{bmatrix}
$$

The communication classes are $C_1 = \{1, 3, 4\}$ and $C_2 = \{2, 5\}$. For any $i \in C_1$ we have $C_{ij} = 1$ for all $j \in C_1$ and $C_{ij} = 0$ for all $j \notin C_1$. Similarly for $C_2$: For any $i \in C_2$ we have $C_{ij} = 1$ for all $j \in C_2$ and $C_{ij} = 0$ for all $j \notin C_2$.

In the next example (50 clusters) we show that sometimes close inputs produce different but very similar agglomerations. Let $T$ be a map such that the communication classes for $C_T$ are $C_1 = \{1, 18, 7, 49, 50\}$, $C_2 = \{2, 5, 21, 33, 37, 46\}$, $C_3 = \{3, 6, 22, 34, 38, 47\}$, $C_4 = \{4, 8, 9, 39, 48\}$, $C_5 = \{10, 12, 23, 35, 40\}$, $C_6 = \{11, 13, 24, 36, 41\}$, $C_7 = \{14, 15, 16, 17, 25, 26, 27\}$, $C_8 = \{28, 30, 32, 19\}$, $C_9 = \{29, 31, 20\}$, $C_{10} = \{42, 43, 44, 45\}$.

Figure 3 shows the histograms of trajectories starting from $x_0 = 0.039999$ (dark gray) and $x_0 = 0.040001$ (gray).

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References